

Gravitational Lensing as a Mechanism For Effective Cloaking

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In light of the surge in popularity of electromagnetic cloaking devices, we consider whether it is possible to use general relativity to *cloak* a volume of spacetime through gravitational lensing. A metric for such a spacetime geometry is presented, and its geometric and physical implications are explained.

I. INTRODUCTION

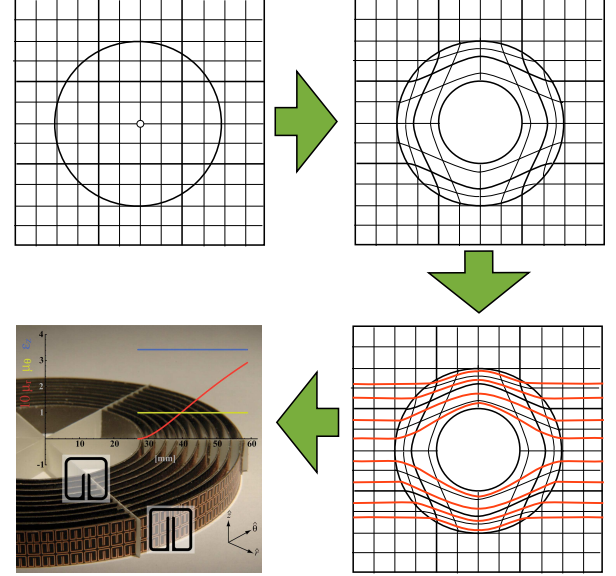
In general relativity, there is a tradition of engineering spacetime geometries with exotic attributes previously seen only in science fiction. Tipler [11] and Morris [7] have introduced time machines; and Alcubierre [2] introduced a warp drive.

In science fiction, one popular conceit is the idea of a cloaking device: a mechanism through which a spaceship could be made undetectable. The revelation that curved spaces can be matched to the electromagnetic properties of a medium has sparked a recent interest in optical cloaking [1, 3–6, 8, 10]. We seek to construct a spacetime geometry which cloaks an interior region from null geodesics.

A. Electromagnetic Cloaking

A lot of interest has recently been awakened in researching cloaking devices. Indeed, since it is now possible to engineer metamaterials with desired exotic electromagnetic properties, a model of a cylindrically symmetric invisibility cloak has even been assembled by Schurig *et al.* [10].

These recent results have come about as a result of a technique known transformation optics which allows us to view refraction equivalently in terms of geodesics on an *virtual curved electromagnetic space*, or in terms of a varying permittivity and permeability in a material [4–6, 8]. Thus, engineering systems with exotic electromagnetic properties becomes a matter of searching for desirable coordinate transformations of flat space [1]. For example, consider the procedure for designing Schurig *et al.*'s cloaking device (see Fig.(1)). We begin with a flat 2-dimensional space, and we draw a circle. We perform a coordinate transformation which expands the point $r = 0$ into a circle of radius r_c . The “straight line” geodesics of the old coordinate system will now follow curved paths which circumnavigate the circle at r_c . We then use the transformed metric to determine the permittivity and



*Diagram from Schurig *et al.* Science 10, Vol. 314 pp.997-980

Figure 1: The procedure for designing an electromagnetic cloaking device.

permeability tensors required for light rays to follow these curves. The result is an object which electromagnetically cloaks objects inside of the r_c circle from the exterior.

This language of geodesics in curved space has opened a dialogue between transformation optics and general relativity [5]. Crudo and O'Brien [3] have even generalized the procedure to consider curved spacetime: given a desired set of geodesics, they determine the metric of the curved spacetime required to generate the geodesics, and then the index of refraction required.

The work done on effective metrics in *virtual* spacetime has inspired us to consider whether cloaking can be achieved as a result of the curvature of *physical* spacetime.

B. Effective Cloaking in Gravity

An optical cloaking system must satisfy two criteria: firstly, an external congruence of light rays which enter

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the system must exit the system undistorted; secondly, these light rays are prevented from penetrating an internal volume. If we generalize these criteria to general relativity, the light rays become null geodesics, and consequently the interior volume is causally isolated from the exterior spacetime.

In lieu of causally isolating the interior volume, let us broaden our definition for cloaking. If a spacetime geometry contains a region through which a congruence of null geodesics can pass undistorted, and if the parameters defining the system can be tuned so that an extended object placed within said region will appear arbitrarily small from the outside; we will refer to the region as *effectively cloaked*. Thus, we could use an effective cloaking geometry to make an object the size of the planet Jupiter appear from the outside to be the size of a pea. In some ways, the mechanism will work in a way opposite of a microscope.

II. GEOMETRIC CLOAKING

A. Interior and Exterior Metric

Our cloaking geometries are a two parameter family of geometries constructed by joining a flat *exterior* spacetime to a curved *interior* spacetime along a spherical hypersurface using the Israel junction conditions [9].

Let the flat exterior have line element:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

where the junction hypersurface is the sphere of constant coordinate radius $r = R$.

Let the interior spacetime have line element:

$$ds^2 = -(\frac{\tilde{r}}{SR})^{2-2S} dt^2 + d\tilde{r}^2 + \frac{1}{S^2} \tilde{r}^2 d\theta^2 + \frac{1}{S^2} \tilde{r}^2 \sin^2 \theta d\phi^2,$$

where $S > 1$ is a constant, and the junction hypersurface is a sphere of constant coordinate radius $\tilde{r} = SR$.

Since the metric is not smooth across this junction, there must be a shell of stress energy confined to the spherical junction hypersurface.

B. Geodesic Congruences

To demonstrate the effective cloaking feature of these geometries, let us consider the trajectories of a congruence of parallel null geodesics incident upon the junction hypersurface from the exterior, where individual geodesics are distinguished by the *impact parameter* b :

$$t = \lambda, r = \sqrt{b^2 + \lambda^2}, \sin \theta = \frac{\lambda}{\sqrt{b^2 + \lambda^2}}, \phi = 0, \quad (1)$$

where λ is the affine time. Due to the spherical symmetry of the system, the trajectory of an arbitrary null geodesic can be described as a rotated member of this congruence.

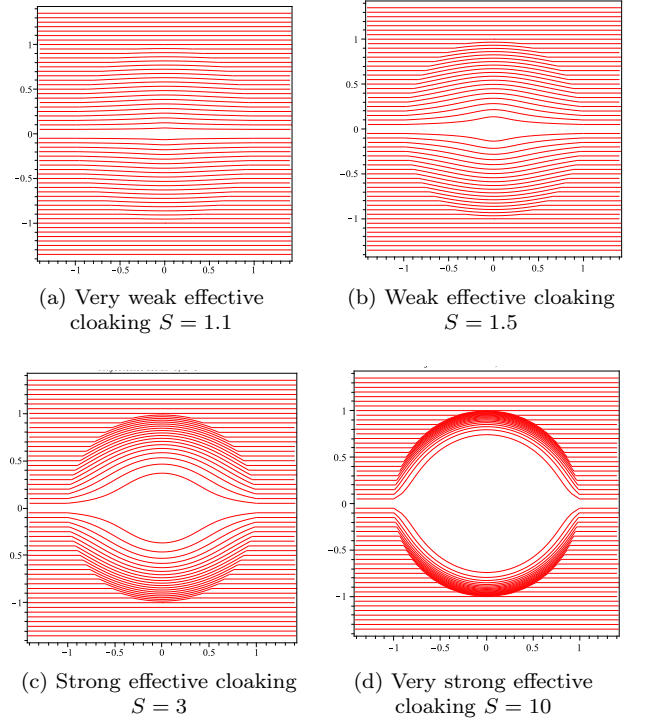


Figure 2: Spatial trajectories of null geodesics moving across $R = 1$ geometries with a variety of S parameters.

The geodesics in this congruence have tangent vectors:

$$\xi_{OUT}^a = [1, \frac{\lambda}{\sqrt{b^2 + \lambda^2}}, \frac{b}{b^2 + \lambda^2}, 0]. \quad (2)$$

Since all of these geodesics lie along a great circle, we need only concern ourselves with the interior null geodesics whose tangent vectors can be written:

$$\xi_{IN}^a = [a_1 (\frac{\tilde{r}}{SR})^\beta, \pm \sqrt{a_1^2 (\frac{\tilde{r}}{SR})^\beta - a_2^2 \frac{S^2}{\tilde{r}^2}}, a_2 \frac{S^2}{\tilde{r}^2}, 0], \quad (3)$$

$$\beta \equiv -2 + 2S$$

where a_1 and a_2 are constants of motion.

We extend the exterior congruence into the interior geometry by matching the projections of the tangent vectors of Eq.(3) and Eq.(2) on the junction hypersurface. The projected tangent vectors on the surface $r = R$ and $\tilde{r} = SR$ are respectively:

$$P_b \equiv h_{ab} \xi_{OUT}^a = [-1, 0, b, 0], \quad (4)$$

$$P_b \equiv h_{ab} \xi_{IN}^a = [-a_1, 0, a_2, 0],$$

where $h_{ab} = g_{ab} - r_a r_b$, and r_a denotes the unit normal to the hypersurface. Thus, the constants of motion for the extended geodesics are: $a_1 = 1$, $a_2 = b$. The geodesics in the parallel congruence Eq. (1), as they move through the interior geometry, will therefore have tangent 4-vectors:

$$\xi_{IN}^a = [(\frac{\tilde{r}}{SR})^{-2+2S}, \sqrt{(\frac{\tilde{r}}{SR})^{-2+2S} - b^2 \frac{S^2}{\tilde{r}^2}}, b \frac{S^2}{\tilde{r}^2}, 0].$$

Thus, the spatial trajectory of the geodesics will have tangents satisfying:

$$\frac{\partial \tilde{r}(\theta)}{\partial \theta} = \frac{1}{b} \frac{\tilde{r}^2}{S^2} \sqrt{\left(\frac{\tilde{r}}{SR}\right)^{-2+2S} - b^2 \frac{S^2}{\tilde{r}^2}}. \quad (5)$$

Note also that the impact parameter b can be re-written in terms of an angle of incidence θ_i , describing the angle at which an external geodesic meets the junction hypersurface: $b \equiv R \cos \theta_i$.

The differential equation Eq.(5) can be solved explicitly through the transformation: $\tilde{r} = RS \left(\frac{1}{F(\theta)}\right)^{\frac{1}{S}}$, where $F(\theta)$ must satisfy:

$$\frac{\partial F}{\partial \theta} = -\frac{1}{b} R \sqrt{1 - \frac{b^2}{R^2} F^2(\theta)}.$$

The unique solution to this differential equation is:

$$F = \frac{R}{b} \cos \theta,$$

and thus the trajectory through the interior must satisfy:

$$\tilde{r}(\theta) = RS \left(\frac{b}{R \cos \theta}\right)^{\frac{1}{S}} = RS \left(\frac{\cos \theta_i}{\cos \theta}\right)^{\frac{1}{S}}. \quad (6)$$

Note that all geodesics in this congruence have a radial turning point halfway across the interior (at $\theta = 0$): $\frac{\partial}{\partial \theta} \tilde{r}(\theta)|_{\theta=0} = 0$. Thus, the trajectories of the geodesics in our congruence are symmetric in reflections along the line $\theta = 0$, and the null congruence will emerge from the interior geometry undeformed.

The family of cloaking geometries have two free parameters: the radius of the junction surface $r = R$, and the value of the parameter $S > 0$. The larger the value of S is, the larger the effect of geodesics splaying away from the center (see Fig. (2)).

Since the image of the star field behind an astronomical object with this spacetime geometry will not be distorted as the photons pass through, we refer to this geometry as an effective cloaking geometry.

III. PROPERTIES OF THE CLOAKING GEOMETRY

A. Reducing the shadow of an object

When we optically gauge the size of an object, we usually do so by looking at its projected area on hypersurfaces normal to a set of null curves which span the space between the observer and the object. In other words, we look at the object's shadow with respect to a set of (preferably parallel) null geodesics (see Fig.(3)). Given enough information, we deduce the object's volume based on the area of this shadow.

For example, imagine that we have been given a metallic sphere, of radius r_{sphere} , and a congruence of parallel

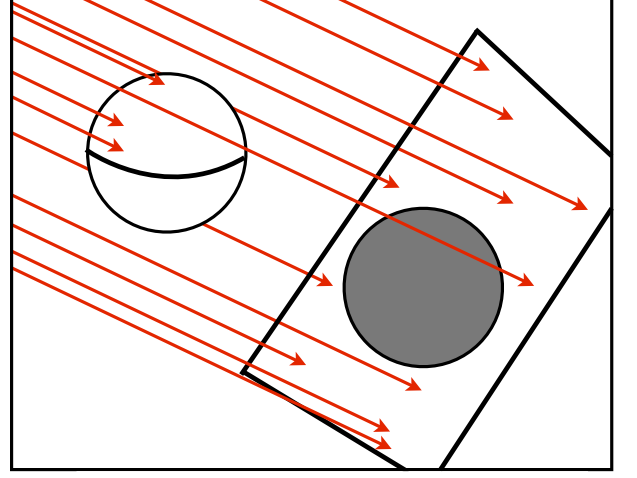


Figure 3: We describe the size of an object with respect to a congruence of geodesics. The object eclipses some of the curves, resulting in a shadow whose area depends on the size of the object, its orientation, and the congruence itself.

null geodesics (called the *background image*) which are partially eclipsed by the sphere. For all orientations, the sphere will cast a shadow of area $A_{sphere} = \pi r_{sphere}^2$ in the background image.

What would this area be if we were to place our metallic sphere at the center of a cloaking geometry? We can determine the area of the shadow, as seen from the outside, by looking at the range of impact parameters b for null geodesics which will be eclipsed by the ball.

In the interior coordinates, our ball will have a radius $\tilde{r}_{sphere} = \frac{r_{sphere}}{S}$. Eq.(6) dictates the trajectory of the geodesics in our background image congruence. These geodesics' minimum radius are at angle $\theta = 0$. Thus, given R and S , the geodesics whose impact parameter b satisfies:

$$r_{sphere} \geq R \left(\frac{b}{R}\right)^{\frac{1}{S}}$$

will encounter our spherical obstruction (where we assume $r_{sphere} < R$). Thus, the edge of the shadow will have impact parameter $b = R \left(\frac{r_{sphere}}{R}\right)^S$, and the area of the shadow will be:

$$A_{effective} = \pi R^2 \left(\frac{r_{sphere}}{R}\right)^{2S}.$$

The larger the value of the parameter S , the smaller the shadow of the sphere will be.

Thus, by tuning the value of S , we can use the cloaking geometry to make an object appear arbitrarily small from the outside.

B. Stress Energy Source

The stress energy tensor of this geometry has two components: the stress-energy generating the interior geometry, and the stress-energy shell which lies on the junction surface:

$$T_{ab} = T_{ab}^{(IN)} + T_{ab}^{\Sigma}.$$

From the Einstein equation, stress energy tensor of the interior geometry is:

$$T_{ab}^{(IN)} = \frac{(S-1)^2}{8\pi} \begin{bmatrix} \frac{(S+1)(SR)^{2S-2}}{\tilde{r}^{2S}(S-1)} & 0 & 0 & 0 \\ 0 & \frac{(S+3)}{\tilde{r}^{2(1-S)}} & 0 & 0 \\ 0 & 0 & \frac{1}{S^2} & 0 \\ 0 & 0 & 0 & \sin^2\theta \frac{1}{S^2} \end{bmatrix}.$$

The fact that null geodesics splay away from the center (in the interior geometry) indicates that the null convergence condition is not satisfied, and thus the null energy condition is violated. We can confirm this explicitly using geodesics of the form Eq. (3):

$$T_{ab}^{(IN)} \xi_{IN}^a \xi_{IN}^b = \frac{2(S-1)}{8\pi} \frac{1}{\tilde{r}^4} (a_2^2(S+1)S^2 - (\frac{1}{SR})^{-2+2S} a_1^2 \tilde{r}^{2S}). \quad (7)$$

since parameters a_1 and a_2 are free, we can choose $a_1 = 1$, $a_2 = 0$, demonstrating that the violation.

The Ricci scalar in the interior is explicitly:

$$R = \frac{6(S-1)}{\tilde{r}^2},$$

demonstrating the existence of a central curvature singularity.

The nonzero components of the second fundamental form on the interior of the junction sphere are:

$$K_{tt}^{(IN)} = \frac{S-1}{RS}, K_{\theta\theta}^{(IN)} = \frac{R}{S}, K_{\phi\phi}^{(IN)} = \frac{R \sin^2(\theta)}{S},$$

and from the exterior:

$$K_{tt}^{(OUT)} = 0, K_{\theta\theta}^{(OUT)} = R, K_{\phi\phi}^{(OUT)} = R \sin^2(\theta).$$

The Israel junction conditions dictate that the discontinuity in the second fundamental form describes a shell of matter located at the junction. A 3-tensor on the hypersurface is defined:

$$S_{ij} \equiv \frac{1}{8\pi} ([K_{ij}] - [K]h_{ij}) = \frac{1}{8\pi} \left(\frac{S-1}{S} \right) \begin{bmatrix} \frac{2}{R} & 0 & 0 \\ 0 & -2R & 0 \\ 0 & 0 & -2R \sin^2\theta \end{bmatrix}. \quad (8)$$

This is used to write the energy density of the shell of matter lying on the junction hypersurface:

$$T_{ab}^{\Sigma} \equiv \delta(\ell) S_{ij} e_a^i e_b^j = -2R \frac{\delta(\ell)}{8\pi} \left(\frac{S-1}{S} \right) \begin{bmatrix} -\frac{1}{R^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2\theta \end{bmatrix}, \quad (9)$$

where ℓ is the length along a congruence normal to the junction, and $\ell \equiv 0$ at the junction.

C. Redshifting

Let us consider a family of stationary timelike observers sitting at constant radii with 4-vectors:

$$T^a = [(\frac{\tilde{r}}{SR})^{S-1}, 0, 0, 0],$$

and look at the redshifts of null geodesics passing between them. In the flat exterior, let us denote the 4-momentum of a photon P^a and the energy of a photon measured by a stationary observer to be γ_i :

$$P_{OUT}^a T^b g_{ab} = -\gamma_i.$$

As the photon travels into the cloaking geometry, the stationary observer measures the photon energy to be:

$$\gamma_f = \gamma_i \left(\frac{\tilde{r}}{SR} \right)^{-1+S}.$$

Thus, the photon is redshifted as it moves towards the center, and blueshifted as it moves outwards.

IV. CONCLUSION

We have presented a two parameter family of spacetimes which demonstrate the effective cloaking of objects placed within them as a result of the gravitational lensing. The analytic trajectory of the null geodesics moving through the cloaking geometry is known, and it is shown that initially parallel geodesics entering the cloaking geometry will splay away from the center, re-converge, and then reemerge in their original, parallel configuration.

The description of this geometry as *effectively cloaking* derives from the way in which an object placed at its center will eclipse fewer null geodesics than it would in flat spacetime. This is due to the splaying of the null geodesics away from the center and around the object. Thus, we use the geometry to make an object appear arbitrarily small from the outside.

Several attributes of this geometry make it arguably physically unrealizable. Firstly, the matter used to construct it must violate the null (and thus the weak and dominant) energy condition. Secondly, this geometry requires an infinitesimally narrow shell of stress-energy to

transition between interior and exterior geometries, and it is unclear what effect allowing a transition of finite width will have on the cloaking properties. The requirement for exotic energy is, however, the same shortcoming found in traversable wormholes and warp drive spacetimes.

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